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Smirnov's integrals and the quantum Knizhnik–Zamolodchikov equation of level 0

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Abstract. We study the quantum Knizhnik–Zamolodchikov equation of level 0 associated with the spin 1/2 representation of $U_q(\widehat{\mathfrak{sl}_2})$. We find an integral formula for solutions in the case of an arbitrary total spin and $|q| < 1$. In the formula, different solutions can be obtained by taking different integral kernels with the cycle of integration being fixed.

1. Introduction

In this paper, following [1], we will give an integral formula for solutions to the quantum Knizhnik–Zamolodchikov (KZ) equation [2] for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ when the spin is 1/2, the level is 0 and $|q| < 1$.

In [1], Smirnov gave an integral formula for the form factors of the sine–Gordon model. His method involved solving a system of difference equations for a vector-valued function in N variables $(\beta_1, \ldots, \beta_N)$, which takes values in the N-fold tensor product of the spin 1/2 representation $\mathbb{C}^2$ of $U_q(\widehat{\mathfrak{sl}_2})$. The total space $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ splits into the subspaces of fixed total spins $(l-n)/2$ where $l+n=N$ and $0 \leqslant n \leqslant N$. In [1], an integral formula was given for the case $n = l = N/2$ (N even) and $|q| = 1$. In a private communication to the present authors, Smirnov showed the modified formula for the case $|q| < 1$, which is given at the end of section 4 of the present paper. Our main contribution is to generalize Smirnov's formula to the case of an arbitrary total spin.

Before going into the details, let us discuss several points concerning the quantum Knizhnik–Zamolodchikov equation and the integral formulae. We are largely indebted to Smirnov for discussions on this matter.

The KZ equation was introduced in [3] as the master equation for the correlation functions of the conformal field theories with gauge symmetries, i.e. the Wess–Zumino–Witten model. In [4], it was studied by using the representation theory of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. The main observation in these developments is that the local operators in the Wess–Zumino–Witten model are realized as the intertwiners of the highest-weight representations (they are called the vertex operators), and the correlation functions which are given as the matrix elements of the products of the vertex operators with respect to the highest-weight vectors, satisfy the KZ equation. The components of the intertwiners belong to finite-dimensional representations of the affine Lie algebras, e.g., the spin 1/2 representation $\mathbb{C}^2$ of $\widehat{\mathfrak{sl}_2}$. Such representations are parametrized by the complex spectral parameters. The correlation functions, therefore, take their values in the tensor products of finite-dimensional representations and depend on the spectral parameters associated with them, say $(\beta_1, \ldots, \beta_N)$.

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This similarity between the KZ equation and the form-factor equation led to the introduction of the quantum KZ equation by Frenkel and Reshetikhin. The representation theory of the affine Lie algebras is replaced by that of the quantum affine algebras, and the system of differential equations is replaced by that of difference equations.

Compared with the quantum KZ equation of Frenkel and Reshetikhin, Smirnov's equation for the sine–Gordon model is special for the following two reasons. First, the former contains a complex parameter; the level of the highest-weight representations. In the latter, the level is set to 0. In this paper, we stick to the level 0 case as in Smirnov's work.

Second, the quantum KZ equation admits a choice of a diagonal operator, which acts on the tensor component whose spectral parameter undergoes a shift (see (2.5)). Smirnov's equation chooses the identity operator for that. In our integral formula, the said operator is not the identity in general, but one fine-tuned according to the total spin $(l-n)/2$. Since Smirnov considered only the case $l=n$, the fine tuning was trivial. To be precise, the case $l=n\pm 2$ is also studied in [1]. However, since it is only in the rational limit, the operator again reduces to the identity.

We now discuss the solutions and the integral formulae.

The analytic structures of the solutions to the quantum KZ equation and the ordinary one differ. The former admits solutions which are meromorphic in the variables $(\beta_1,\dots,\beta_N)$, i.e., the additive spectral parameters. The latter, in general, forces its solutions to exhibit branches. Namely, the monodromy structures of the solutions are very different. In fact, it is more important to consider the braid relations for the solutions, i.e. the behaviour of the solutions when two of the variables are interchanged. In [4], the braid representations for the KZ equation were nicely deduced from the commutation relation for the vertex operators. As in [2, 5], the quantum vertex operators enjoy a similar commutation relation. The difference in these two commutation relations is that in the former the coefficients are independent of the spectral parameters but in the latter they are not.

The said difference is closely related to the following fact. Being a holonomic system of differential equations, the space of the solutions to the KZ equation is finite-dimensional, while the quantum KZ equation admits the linearity with the coefficients in the ring of quasi-constants, i.e. functions invariant under the relevant shifts of the spectral parameters. Therefore, the braiding property of the solutions to the KZ equation is unique up to finite-dimensional similarity transformations while it is quite ambiguous for the quantum KZ equation. Nevertheless, the commutation relations of the quantum vertex operators imply that the braid relation with the R-matrix in the coefficients (we call it the R-matrix symmetry, see (2.3)) is indeed compatible with the difference equation. For example, the solutions obtained from the quantum vertex operators satisfy this property. Smirnov suggests that we can utilize the freedom of the choice of the quasi-constants to choose this particular braid relation for the solutions. Thus, we demand R-matrix symmetry in constructing the integral formulae. Then, the vector-valued unknown function reduces to a single function, and the system of difference equations reduces to a certain deformed cyclicity (2.4).

Once the equation is thus reduced, we can construct the integral formula as follows. Let us explain our formula in the case $n<l$. We find it convenient to employ the multiplicative spectral parameters $(z_1,\dots,z_N)$, where $\beta_i\propto\log z_i$. Following Smirnov, we choose an integral kernel $\Psi(x_1,\dots,x_n|z_1,\dots,z_N)$. It has simple poles at $x_\mu=z_jq^{\pm(1+4k)}$ $(k=0,1,\dots)$, and satisfies certain quasi-periodicities with respect to the shifts of the variables $x_\mu\to x_\mu q^4$ and $z_j\to z_jq^4$. In fact, the choice of such an integral kernel is not unique because the solutions to the quasi-periodicity conditions are not unique. Smirnov's idea is that this freedom in the integrand corresponds to the freedom of the

solutions up to quasi-constants. The total integrand contains a fixed rational function $F = F(x_1, \ldots, x_n|z_1, \ldots, z_N)$ other than the said kernel. It is a polynomial in x_μs and homogeneous in x_μs and z_js. We consider the total integral as an integral transform of this rational function in terms of the integral kernel, fixing appropriately the cycles for the integrations with respect to x_μs. It is easily shown that if F is a polynomial of degree greater than $N-1$ in a variable x_μ, then it is reducible to a lower-degree polynomial modulo; a certain 'total difference' which vanishes after integration with respect to x_μ. Using this fact, we can write down a system of algebraic relations for F as a sufficient condition for the deformed cyclicity. These conditions are also necessary if we assume that there are no other lower-degree relations of the kind mentioned above. Thus, we find F satisfying them. The details will be given in the subsequent sections.

Lastly, we discuss the related works on the integral formulae for the quantum KZ equations. In [6,7], solutions by Jackson-type integrals are obtained. Their formulae are, in principle, valid for the general level, as opposed to our integral formula restricted to level 0. On the other hand, the problem of choosing the cycles for Jackson-type integrals, which accommodate the freedom of the solutions, is not well studied. In particular, the choice of the cycles that leads to R-matrix symmetry is totally unclear.

A different type of integral formula was obtained in [8] by using the Frenkel–Jing bosonization of the level 1 highest-weight representations [9]. Though the level can be chosen arbitrarily, it gives only one particular solution for each level. Since the relation between the formulae in this paper and those in [8] are not yet clear, we do not discuss this matter further.

The rest of the paper is organized as follows. In section 2, we formulate the difference equation in terms of a single component of the vector. In section 3, we solve the algebraic relation for F in the case $n = 1$. In section 4 we solve the general case recursively.

2. Difference equations

The purpose of this section is to formulate the problem, thereby fixing our notations.

Let us begin by recalling the standard trigonometric R-matrix $R(z) \in \mathrm{End}(V \otimes V)$ associated with $V = \mathbb{C}^2$. Fix a complex number q such that $0 < |q| < 1$. The matrix $R(z)$ is specified by giving the matrix elements relative to the standard basis $v_+, v_- \in V$

$$R(z) v_{\varepsilon_1'} \otimes v_{\varepsilon_2'} = \sum_{\varepsilon_1, \varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2} R(z)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'}.$$

The non-zero entries are

$$R(z)^{++}_{++} = R(z)^{--}_{--} = a(z)$$

$$R(z)^{+-}_{+-} = R(z)^{-+}_{-+} = b(z)$$

$$R(z)^{+-}_{-+} = R(z)^{-+}_{+-} = c(z)$$

where

$$a(z) = 1 \qquad b(z) = \frac{(1-z)q}{1-zq^2} \qquad c(z) = \frac{(1-q^2)\sqrt{z}}{1-zq^2}.$$

In what follows, we shall work with the tensor product $V^{\otimes N}$. Following the usual convention, we let $R_{jk}(z)$ $(j \neq k)$ signify the operator on $V^{\otimes N}$ acting as $R(z)$ on the

(j,k)th tensor components and as the identity on the other components. In particular we have $R_{kj}(z) = P_{jk}R_{jk}(z)P_{jk}$, where $P \in \mathrm{End}(V \otimes V)$ stands for the transposition $P(x \otimes y) = y \otimes x$.

The main properties of $R(z)$ are the Yang–Baxter equation

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2) \qquad (2.1)$$

and the unitarity relation

$$R_{12}(z_1/z_2)R_{21}(z_2/z_1) = 1. \qquad (2.2)$$

The equations that concern us with in this paper are the following ones for a function $G(z_1, \cdots, z_N)$ with values in $V^{\otimes N}$:

(i) R-matrix symmetry

$$P_{j\,j+1}G(\ldots, z_{j+1}, z_j, \ldots) = R_{j\,j+1}(z_j/z_{j+1})G(\ldots, z_j, z_{j+1}, \ldots) \qquad (1 \leqslant j \leqslant N-1). \qquad (2.3)$$

(ii) Deformed cyclicity

$$P_{12}\ldots P_{N-1\,N}G(z_2, \ldots, z_N, z_1q^{-4}) = D_1G(z_1, \ldots, z_N). \qquad (2.4)$$

In (2.4), D_1 is an operator acting on the first component as $D = \mathrm{diag}(\delta_+, \delta_-)$, the entries of which will be specified below, and acting as the identity on the other components. Here a remark is in order about the precise meaning of these equations. The functions we consider throughout this article are not necessarily single valued in z_j but are meromorphic in the variable $\log z_j$. Accordingly, the shift $z_j \to z_jq^{-4}$, as in (2.4), is understood to mean $\log z_j \to \log z_j - 4\log q$.

The equations (2.3) and (2.4) with $D = 1$ appeared in Smirnov's works on the form factors of massive integrable field theories [1]. As was pointed out in [10], they imply the quantum KZ equation of level 0 [2]

$$\begin{aligned} G(z_1, \ldots, z_jq^4, \ldots, z_N) &= R_{j-1\,j}(z_{j-1}/z_jq^4)^{-1}\ldots R_{1\,j}(z_1/z_jq^4)^{-1}D_j^{-1} \\ &\quad\times R_{j\,N}(z_j/z_N)\ldots R_{j\,j+1}(z_j/z_{j+1})G(z_1, \ldots, z_j, \ldots, z_N). \end{aligned} \qquad (2.5)$$

These equations have a $\mathbb{Z}_2$-symmetry which means that if $G(z_1, \ldots, z_N)$ solves (2.3), (2.4) then $\tilde{G}(z_1, \ldots, z_N) = \sigma^x \otimes \ldots \otimes \sigma^x G(z_1, \ldots, z_N)$ solves the same system wherein δ_+ and δ_- are interchanged and where σ^x is given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the following, we set $\tau = q^{-1}$. We define the components of G by

$$G(z_1, \ldots, z_N) = \sum_{\varepsilon_j=\pm} v_{\varepsilon_1} \otimes \ldots \otimes v_{\varepsilon_N}\left(\prod_{\varepsilon_j<0}\sqrt{z_j}\right)G^{\varepsilon_1\ldots\varepsilon_N}(z_1, \ldots, z_N). \qquad (2.6)$$

Then the equation (2.3) (resp. (2.4)) reads as (2.7) (2.8) (resp. (2.9)):

$$G^{\cdots \overset{j}{\varepsilon} \overset{j+1}{\varepsilon} \cdots}(\dots, z_j, z_{j+1}, \dots) = G^{\cdots \overset{j}{\varepsilon} \overset{j+1}{\varepsilon} \cdots}(\dots, z_{j+1}, z_j, \dots) \tag{2.7}$$

$$\begin{aligned} G^{\cdots \overset{j}{+} \overset{j+1}{-} \cdots}(\dots, z_j, z_{j+1}, \dots) &= \frac{z_j - z_{j+1}\tau^2}{(z_j - z_{j+1})\tau} G^{\cdots \overset{j}{-} \overset{j+1}{+} \cdots}(\dots, z_{j+1}, z_j, \dots) \\ &\quad - \frac{(1-\tau^2) z_j}{(z_j - z_{j+1})\tau} G^{\cdots \overset{j}{-} \overset{j+1}{+} \cdots}(\dots, z_j, z_{j+1}, \dots) \end{aligned} \tag{2.8}$$

$$G^{\varepsilon_2 \dots \varepsilon_N \varepsilon_1}(z_2, \dots, z_N, z_1\tau^4) = \delta_{\varepsilon_1} G^{\varepsilon_1 \varepsilon_2 \dots \varepsilon_N}(z_1, z_2, \dots, z_N). \tag{2.9}$$

The factor $\prod_{\varepsilon_j<0} \sqrt{z_j}$ in (2.6) is chosen so that the coefficients in these equations are free from square root symbols. Note that the singularity at $z_j = z_{j+1}$ in (2.8) is spurious. Equations (2.7)–(2.9) split into blocks, each involving components such that

$$n = \sharp\{j \mid \varepsilon_j = -\} \qquad l = \sharp\{j \mid \varepsilon_j = +\} \qquad (n + l = N)$$

are fixed. Because of the $\mathbb{Z}_2$-symmetry we may assume $n \leqslant l$ without loss of generality.

Consider the extreme component

$$G^{\overbrace{-\cdots-}^{n}\overbrace{+\cdots+}^{l}}(z_1, \dots, z_N) = H(z_1, \dots, z_n \mid z_{n+1}, \dots, z_N). \tag{2.10}$$

Because of (2.7), this function is symmetric separately in the variables $(z_1, \dots, z_n)$ and $(z_{n+1}, \dots, z_N)$. Equation (2.8) tells that all the components with fixed n, l are uniquely determined from H. Conversely, given any such H, the Yang–Baxter equation guarantees that (2.3) can be solved consistently under the condition (2.10).

Remark. An explicit way of reconstructing G from H is described in [1]. The procedure is as follows. Define the operator $B(z_1, \dots, z_N|t) \in \mathrm{End}(V^{\otimes N})$ by

$$R_{1\,N+1}(z_1/t) \dots R_{N\,N+1}(z_N/t) = \begin{pmatrix} A(z_1, \dots, z_N|t) & B(z_1, \dots, z_N|t) \\ C(z_1, \dots, z_N|t) & D(z_1, \dots, z_N|t) \end{pmatrix}.$$

Here the 2×2 matrix structure is defined relative to the base $v_\pm$ of the $(N+1)$th tensor component of $V^{\otimes(N+1)}$. For $\alpha = (\alpha_1, \dots, \alpha_N)$ with $\alpha_i = \pm$, set $J^\alpha_\pm = \{j \mid \alpha_j = \pm\}$ and

$$w_\alpha(z_1, \dots, z_N) = \prod_{m \in J^\alpha_-} B(z_1, \dots, z_N | z_{\alpha_m})\Omega$$

$$\Omega = v_+ \otimes \dots \otimes v_+ \qquad \in V^{\otimes N}.$$

Then

$$G(z_1, \dots, z_N) = \sum_\alpha w_\alpha(z_1, \dots, z_N) H(\{z_m\}_{m \in J^\alpha_-} \mid \{z_p\}_{p \in J^\alpha_+}) \prod_{\substack{m \in J^\alpha_- \\ p \in J^\alpha_+}} \frac{1}{b(z_p/z_m)} \prod_{m \in J^\alpha_-} \sqrt{z_m}.$$

Under the relations (2.7) and (2.8), it is sufficient to consider the remaining relation (2.9) for two cases with $\varepsilon_1 = +$ and $\varepsilon_1 = -$, e.g., $(\varepsilon_1, \dots, \varepsilon_N) = (+\underbrace{-\dots-}_{n}\underbrace{+\dots+}_{l-1})$ and

$(\underbrace{-\ldots-}_{n-1}\underbrace{+\ldots+}_{l}-)$. Solving (2.8) for the corresponding components in terms of (2.10), we find that the original system (2.3), (2.4) is equivalent to the following for the single function $H(z_1,\ldots,z_n\,|\,z_{n+1},\ldots,z_N)$, satisfying the said symmetry condition,

$$\delta_+^{-1}H(z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4)=\prod_{j=2}^{n+1}\frac{z_1-z_j\tau^2}{(z_1-z_j)\tau}H(z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1)$$
$$-\sum_{j=2}^{n+1}\frac{(1-\tau^2)z_1}{(z_1-z_j)\tau}\prod_{\substack{k=2\\k\neq j}}^{n+1}\frac{z_j-z_k\tau^2}{(z_j-z_k)\tau}H(z_1,z_2,\overset{j}{\hat{\ldots}},z_{n+1}|z_j,z_{n+2},\ldots,z_N) \tag{2.11}$$

$$\delta_-\tau^{-2}H(z_1\tau^{-4},z_2,\ldots,z_n|z_{n+1},\ldots,z_N)=\prod_{j=n+1}^{N}\frac{z_1-z_j\tau^{-2}}{(z_1-z_j)\tau^{-1}}H(z_1,\ldots,z_n|z_{n+1},\ldots,z_N)$$
$$-\sum_{j=n+1}^{N}\frac{(1-\tau^{-2})z_j}{(z_1-z_j)\tau^{-1}}\prod_{\substack{k=n+1\\k\neq j}}^{N}\frac{z_j-z_k\tau^{-2}}{(z_j-z_k)\tau^{-1}}H(z_2,\ldots,z_n,z_j|z_1,z_{n+1},\overset{j}{\hat{\ldots}},z_N). \tag{2.12}$$

In what follows we tune $\delta_+=\tau^{-n}$ and $\delta_-=\tau^{-l}$. We wish to find an integral formula of the form

$$H(z_1,\ldots,z_N)=(S_{nN}F)(z_1,\ldots,z_N) \tag{2.13}$$

where S_{nN} stands for the integral transform

$$(S_{nN}F)(z_1,\ldots,z_N)=\prod_{\mu=1}^{n}\oint_C \mathrm{d}x_\mu\, F(x_1,\ldots,x_n|z_1,\ldots,z_N)\Psi(x_1,\ldots,x_n|z_1,\ldots,z_N). \tag{2.14}$$

The notation is explained below.

The kernel Ψ has the form

$$\Psi(x_1,\ldots,x_n|z_1,\ldots,z_N)=\vartheta(x_1,\ldots,x_n|z_1,\ldots,z_N)\prod_{\mu=1}^{n}\prod_{j=1}^{N}\psi\left(\frac{x_\mu}{z_j}\right)$$

where

$$\psi(z)=\frac{1}{(zq;q^4)_\infty(z^{-1}q;q^4)_\infty}\qquad (z;p)_\infty=\prod_{n=0}^{\infty}(1-zp^n).$$

For the function ϑ we assume:

- that it is anti-symmetric and holomorphic in the $x_\mu\in\mathbb{C}\backslash\{0\}$;
- that it is symmetric and meromorphic in the $\log z_j\in\mathbb{C}$;
- that it has the transformation property

$$\vartheta(x_1,\ldots,x_n|z_1,\ldots,z_j\tau^4,\ldots,z_N)=\vartheta(x_1,\ldots,x_n|z_1,\ldots,z_N)\prod_{\mu=1}^{n}\frac{-z_j\tau}{x_\mu}$$
$$\vartheta(x_1,\ldots,x_\mu\tau^4,\ldots,x_n|z_1,\ldots,z_N)=\vartheta(x_1,\ldots,x_n|z_1,\ldots,z_N)\prod_{j=1}^{N}\frac{-x_\mu\tau}{z_j}. \tag{2.15}$$

The function ϑ is otherwise arbitrary and the choice of ϑs corresponds to that of solutions. The transformation property of ϑ implies

$$\frac{\Psi(x_1,\ldots,x_n|z_1,\ldots,z_j\tau^4,\ldots,z_N)}{\Psi(x_1,\ldots,x_n|z_1,\ldots,z_N)}=\prod_{\mu=1}^{n}\frac{x_\mu-z_j\tau}{x_\mu-z_j\tau^3} \tag{2.16}$$

$$\frac{\Psi(x_1,\ldots,x_\mu\tau^4,\ldots,x_n|z_1,\ldots,z_N)}{\Psi(x_1,\ldots,x_n|z_1,\ldots,z_N)}=\tau^{-2N}\prod_{j=1}^{N}\frac{x_\mu-z_j\tau^{-1}}{x_\mu-z_j\tau^{-3}}. \tag{2.17}$$

The integration $\oint_C \mathrm{d}x_\mu$ is along a simple closed curve C, oriented anti-clockwise, which encircles the points $z_j\tau^{-1-4k}(1\leqslant j\leqslant N, k\geqslant 0)$ but not $z_j\tau^{1+4k}(1\leqslant j\leqslant N, k\geqslant 0)$. Finally, F has the form

$$F(x_1,\ldots,x_n|z_1,\ldots,z_N)=\frac{\Delta^{(nl)}(x_1,\ldots,x_n|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)}{\prod_{j=1}^{n}\prod_{i=n+1}^{N}(z_i-z_j\tau^2)} \tag{2.18}$$

where $\Delta^{(nl)}$ is a certain homogeneous polynomial to be determined; antisymmetric in the variables $(x_1,\ldots,x_n)$ and symmetric in the variables $(z_1,\ldots,z_n)$ and $(z_{n+1},\ldots,z_N)$, separately.

In the following sections we shall find a formula $\Delta^{(nl)}$.

3. The case $n=1<l$

In this section we find $\Delta=\Delta^{(1l)}$ for $l>1$. The result will be given in (3.4). First, we prepare some lemmas.

Lemma 3.1. Let f be a polynomial in x and set

$$\begin{aligned}\tilde f(x|z_1,\ldots,z_N)={}&\frac{x-z_1\tau}{x-z_1\tau^3}f(x|z_1\tau^4,z_2,\ldots,z_N)+\frac{z_1(-\tau)^{4-N}}{x\prod_{j=2}^{N}(z_j-z_1\tau^2)}\\&\times\left\{\frac{\prod_{j=1}^{N}(x-z_j\tau)}{x-z_1\tau^3}-\tau^{2(N-2)}\prod_{j=2}^{N}(x-z_j\tau^{-1})\right\}f(z_1\tau^3|z_1\tau^4,z_2,\ldots,z_N).\end{aligned} \tag{3.1}$$

Then we have

$$(S_{1N}f)(z_1\tau^4,z_2,\ldots,z_N)=(S_{1N}\tilde f)(z_1,\ldots,z_N) \tag{3.2}$$

where the left-hand side is the analytic continuation of $(S_{1N}f)(z_1,\ldots,z_N)$ in the variable z_1.

Proof. When the integral (3.2) is analytically continued from z_1 to $z_1\tau^4$, the poles of the integrand move from $x = \dots, z_1\tau^{-5}, z_1\tau^{-1}, z_1\tau, z_1\tau^5, \dots$ to $x = \dots, z_1\tau^{-1}, z_1\tau^3, z_1\tau^5, z_1\tau^9, \dots$. In particular, the pole which moves from $x = z_1\tau^{-1}$ to $x = z_1\tau^3$ crosses the original contour C. Using (2.16), we obtain

$$(S_{1N}f)(z_1\tau^4, z_2, \dots, z_N) = \left(\oint_C + 2\pi \mathrm{i}\, \mathrm{Res}_{x=z_1\tau^3}\right) \mathrm{d}x \\ \times f(x|z_1\tau^4, z_2, \dots, z_N)\frac{x - z_1\tau}{x - z_1\tau^3}\Psi(x|z_1, \dots, z_N).$$

Set

$$r(x) = \frac{z_1\tau^3}{x}\prod_{j=2}^{N}\frac{x - z_j\tau}{z_1\tau^3 - z_j\tau}.$$

Then it has zeros at $x = z_j\tau$ $(j = 2, \dots, N)$ and is equal to 1 at $x = z_1\tau^3$ and therefore the residue at $x = z_1\tau^3$ can be replaced by the difference of two integrals as follows.

$$2\pi \mathrm{i}\, \mathrm{Res}_{x=z_1\tau^3}\ \mathrm{d}x\ f(x|z_1\tau^4, z_2, \dots, z_N)\frac{x - z_1\tau}{x - z_1\tau^3}\Psi(x|z_1, \dots, z_N)$$
$$= \left\{\oint_{\tau^4 C} - \oint_C\right\} \mathrm{d}x\ f(z_1\tau^3|z_1\tau^4, z_2, \dots, z_N)\frac{x - z_1\tau}{x - z_1\tau^3} r(x)\Psi(x|z_1 \dots, z_N)$$
$$= -\oint_C \mathrm{d}x\left\{\frac{x - z_1\tau}{x - z_1\tau^3} r(x) - \tau^{-2(N-2)}\frac{x\tau^4 - z_1\tau}{x\tau^4 - z_1\tau^3}\prod_{j=1}^{N}\frac{x - z_j\tau^{-1}}{x - z_j\tau^{-3}} r(x\tau^4)\right\}$$
$$\times f(z_1\tau^3|z_1\tau^4, z_2, \dots, z_N)\Psi(x|z_1, \dots, z_N).$$

In the second equality we use (2.17). Thus we obtain (3.2). □

Lemma 3.2. If $f(x|z_1, \dots, z_N)$ is a polynomial in x, then there exists $g(x|z_1, \dots, z_N)$; a polynomial in x, of degree less than or equal to $N - 2$, such that $S_{1N}f = S_{1N}g$.

Proof. Since the product $\prod_{j=1}^{N}(x - z_j\tau)$ cancels the simple poles at $x = z_j\tau$ of $\Psi(x|z_1, \dots, z_N)$, we can deform the contour C to $\tau^4 C$ without crossing the poles as follows.

$$\oint_C \mathrm{d}x\, x^{k-1}\prod_{j=1}^{N}(x - z_j\tau)\Psi(x|z_1, \dots, z_N) = \oint_{\tau^4 C} \mathrm{d}x\, x^{k-1}\prod_{j=1}^{N}(x - z_j\tau)\Psi(x|z_1, \dots, z_N)$$
$$= \tau^{2N+4k}\oint_C \mathrm{d}x\, x^{k-1}\prod_{j=1}^{N}(x - z_j\tau^{-1})\Psi(x|z_1, \dots, z_N).$$

This implies that

$$\oint_C \mathrm{d}x\, f(x)\Psi(x|z_1, \dots, z_N) = 0$$

for the polynomials

$$f(x) = x^{k-1}\left\{\prod_{j=1}^{N}(x - z_j\tau) - \tau^{2N+4k}\prod_{j=1}^{N}(x - z_j\tau^{-1})\right\} \qquad k \geqslant 0.$$

From these relations any polynomial is reducible to some g of degree less than $N - 1$. □

We now solve (2.11) in the form $H = S_{1N}F$, where F is given by (2.18) with $n = 1$. Because of (3.2), the left-hand side of (2.11) is equal to $\tau S_{1N}\tilde{F}_1$ where $F_1(x|z_1,\ldots,z_N) = F(x|z_2,\ldots,z_N,z_1)$ and $\tilde{F}_1$ is given by (3.1) with f replaced by F_1. Similarly, the right-hand side of (2.11) is equal to $S_{1N}F_2$ for a certain $\tilde{F}_2$. Because of lemma 3.2, we make an ansatz that the equation $\tau\tilde{F}_1 = \tilde{F}_2$ is satisfied by an F such that Δ is a homogeneous polynomial of degree $N-2$. In fact, this ansatz uniquely determines Δ as we will see shortly.

Define

$$h^{(N)}(x|z_1,\ldots,z_N) := \frac{1}{x}\left\{\prod_{j=1}^{N}(x - z_j\tau) - \tau^{2N}\prod_{j=1}^{N}(x - z_j\tau^{-1})\right\}. \tag{3.3}$$

Equation (2.11) for Δ reads

$$\begin{aligned}
&\frac{1}{(z_2 - z_1\tau^2)\prod_{j=3}^{N}(z_j - z_2\tau^2)}\frac{x - z_1\tau}{x - z_1\tau^3}\Delta(x|z_2|z_3,\ldots,z_N,z_1\tau^4)\\
&\quad + \frac{(-\tau)^{4-N}z_1\Delta(z_1\tau^3|z_2|z_3,\ldots,z_N,z_1\tau^4)}{x(z_2 - z_1\tau^2)^2\prod_{j=3}^{N}(z_j - z_1\tau^2)\prod_{j=3}^{N}(z_j - z_2\tau^2)}\\
&\quad \times\left\{\frac{\prod_{j=1}^{N}(x - z_j\tau)}{x - z_1\tau^3} - \tau^{2(N-2)}\prod_{j=2}^{N}(x - z_j\tau^{-1})\right\}\\
&= \frac{1}{z_2 - z_1}\left\{\frac{1}{\prod_{j=3}^{N}(z_j - z_2\tau^2)}\Delta(x|z_2|z_3,\ldots,z_N,z_1)\right.\\
&\quad \left. - \frac{(1-\tau^2)z_1}{\prod_{j=2}^{N}(z_j - z_1\tau^2)}\Delta(x|z_1|z_2,\ldots,z_N)\right\}.
\end{aligned}$$

Comparing the residues at $z_3 = z_1\tau^2$, we obtain

$$\begin{aligned}
\Delta(x|z_1|z_2, z_1\tau^2, z_4, \ldots, z_N) &= \frac{(-\tau)^{2-N}(x - z_1\tau)}{(1-\tau^2)(z_2 - z_1\tau^2)}\prod_{j=4}^{N}\frac{1}{z_j - z_2\tau^2}h^{(N-2)}(x|z_2, z_4, \ldots, z_N)\\
&\quad \times \Delta(z_1\tau^3|z_2|z_1\tau^2, z_4, \ldots, z_N, z_1\tau^4).
\end{aligned}$$

This determines the restriction of Δ at $z_3 = z_1\tau^2$ up to a constant multiple. Choosing the constant appropriately we have

$$\Delta(x|z_1|z_2, z_1\tau^2, z_4, \ldots, z_N) = (x - z_1\tau)h^{(N-2)}(x|z_2, z_4, \ldots, z_N).$$

Because of the symmetry of $\Delta(x|z_1|z_2,\ldots,z_N)$ with respect to $(z_2,\ldots,z_N)$, we have a similar equation for the restriction at $z_j = z_1\tau^2$ for $2 \leqslant j \leqslant N$. These $N-1$ restrictions uniquely determine the degree $N-2$ polynomial $\Delta(x|z_1|z_2,\ldots,z_N)$

$$\Delta(x|z_1|z_2,\ldots,z_N) = (x - z_1\tau)\sum_{\kappa=0}^{N-3}(1-\tau^{2(N-2-\kappa)})(-\tau)^{\kappa}x^{N-3-\kappa}\sum_{\lambda=0}^{\kappa}(-z_1\tau^2)^{\lambda}\sigma_{\kappa-\lambda}(z_2\ldots z_N) \tag{3.4}$$

where $\sigma_\kappa(a_1,\ldots,a_n)$ denotes the κth elementary symmetric polynomial

$$\prod_{j=1}^{n}(t-a_j)=\sum_{\kappa=0}^{n}(-1)^\kappa\sigma_\kappa(a_1,\ldots,a_n)t^{n-\kappa}.$$

To prove that this Δ actually solves (2.11) and (2.12), we use the following simple facts. Let P, Q be homogeneous rational functions of multi-variables. Suppose that the poles of P, Q are simple and they are contained in a union of k hyperplanes H_j $(1\leqslant j\leqslant k)$. Then, we can conclude $P=Q$ in each of the following:

- the degrees of P, Q are less than or equal to -1 and all the residues of P, Q at H_js coincide.
- the degrees of P, Q are less than or equal to $-m$ and the residues of P, Q coincide at H_j for at least $(k-m+1)$ of them.

4. The general case

We now present our integral formula for the case $n<l$. The case $n=l$ will be discussed at the end of this section. The polynomial $\Delta^{(nl)}$ in (2.18) is given by

$$\Delta^{(nl)}(x_1,\ldots,x_n|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)=\det(A_\lambda^{(nl)}(x_\mu|z_1,\ldots,z_n|z_{n+1},\ldots,z_N))_{1\leqslant\lambda,\mu\leqslant n}. \tag{4.1}$$

The entries of the $n\times n$ matrix $A^{(nl)}$ are defined as follows. Let us introduce the polynomials $\tilde\sigma_\kappa(a_1,\ldots,a_n)$ by

$$\prod_{j=1}^{n}(t-a_j)^{-1}=\sum_{\kappa\geqslant0}\tilde\sigma_\kappa(a_1,\ldots,a_n)t^{-n-\kappa}.$$

For $\lambda, n, l\geqslant 0$, define the following polynomials:

$$f_\lambda^{(nl)}(x|a_1,\ldots,a_n|b_1,\ldots,b_l)=\sum_{\kappa=0}^{l-n+\lambda-2}(1-\tau^{2(l-n+\lambda-1-\kappa)})(-\tau)^\kappa\varphi_{\lambda\kappa}^{(nl)}(a_1,\ldots,a_n|b_1,\ldots,b_l)x^{l-n+\lambda-2-\kappa}$$

where

$$\varphi_{\lambda\kappa}^{(nl)}(a_1,\ldots,a_n|b_1,\ldots,b_l)=\sigma_\kappa(b_1,\ldots,b_l)-\sum_{\lambda\leqslant\alpha\leqslant\beta\leqslant\kappa}(-1)^{\beta-\alpha}\tau^{2\beta}\sigma_{\kappa-\beta}(b_1,\ldots,b_l)\sigma_\alpha(a_1,\ldots,a_n)\tilde\sigma_{\beta-\alpha}(a_1,\ldots,a_n)$$

and

$$g_\lambda^{(n)}(x|a_1,\ldots,a_n)=\sum_{\kappa=0}^{\lambda-2}(1-\tau^{2(\lambda-1-\kappa)})(-\tau)^\kappa\sigma_\kappa(a_1,\ldots,a_n)x^{\lambda-2-\kappa}.$$

Note that $\varphi^{(nl)}_{\lambda\kappa}(a_1,\ldots,a_n|b_1,\ldots,b_l)=\sigma_\kappa(b_1,\ldots,b_l)$ if $\kappa<\lambda$ or $\lambda>n$. For $\lambda\geqslant 0$ and $n<l$, set

$$A^{(nl)}_\lambda(x|a_1,\ldots,a_n|b_1,\ldots,b_l):=\prod_{j=1}^{n}(x-a_j\tau)f^{(nl)}_\lambda(x|a_1,\ldots,a_n|b_1,\ldots,b_l)$$

$$+\tau^{2(l-n+\lambda-1)}\prod_{i=1}^{l}(x-b_i\tau^{-1})g^{(n)}_\lambda(x|a_1,\ldots,a_n). \tag{4.2}$$

This is a homogeneous polynomial of degree $l+\lambda-2$, symmetric with respect to a_js and b_is separately. Notice that

$$A^{(nl)}_\lambda(x|a_1,\ldots,a_n|b_1,\ldots b_l)\text{ is linear with respect to }b_i\text{s}. \tag{4.3}$$

By the construction (4.1) and (4.2), $\Delta^{(nl)}$ is a homogeneous polynomial of degree $n(l-1)+n(n-1)/2$ with the correct symmetries.

Lemma 4.1. The polynomial $A^{(nl)}_\lambda$ obeys the following recursion relation

$$A^{(nl)}_\lambda(x|a_1,\ldots,a_{n-1},a|b_1,\ldots,b_{l-1},a\tau^2)=(x-a\tau)\{A^{(n-1l-1)}_\lambda(x|a_1,\ldots,a_{n-1}|b_1,\ldots,b_{l-1})$$
$$-a\tau^3A^{(n-1l-1)}_{\lambda-1}(x|a_1,\ldots,a_{n-1}|b_1,\ldots,b_{l-1})\}.$$

Proof. It follows from the recursions for $f^{(nl)}_\lambda$ and $g^{(n)}_\lambda$

$$f^{(nl)}_\lambda(x|a_1,\ldots,a_{n-1},a|b_1,\ldots,b_{l-1},a\tau^2)=f^{(n-1l-1)}_\lambda(x|a_1,\ldots,a_{n-1}|b_1,\ldots,b_{l-1})$$
$$-a\tau^3f^{(n-1l-1)}_{\lambda-1}(x|a_1,\ldots,a_{n-1}|b_1,\ldots,b_{l-1})$$

$$g^{(n)}_\lambda(x|a_1,\ldots,a_{n-1},a)=g^{(n-1)}_\lambda(x|a_1,\ldots,a_{n-1})-a\tau g^{(n-1)}_{\lambda-1}(x|a_1,\ldots,a_{n-1}).$$ □

Lemma 4.2. The determinant $\Delta^{(nl)}$ obeys the following recursion relation

$$\Delta^{(nl)}(x_1,\ldots,x_n|a_1,\ldots,a_{n-1},a|b_1,\ldots,b_{l-1},a\tau^2)$$
$$=\prod_{\mu=1}^{n}(x_\mu-a\tau)\sum_{\nu=1}^{n}(-1)^{n+\nu}h^{(N-2)}(x_\nu|a_1,\ldots,a_{n-1},b_1,\ldots,b_{l-1})$$
$$\times\Delta^{(n-1l-1)}(x_1,\overset{\nu}{\hat{\ldots}},x_n|a_1,\ldots,a_{n-1}|b_1,\ldots,b_{l-1}) \tag{4.4}$$

where $h^{(N)}$ is defined in (3.3).

Proof. By the definition we get

$$A^{(nl)}_0(x|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)=0$$

and

$$A^{(nl)}_{n+1}(x|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)=h^{(N)}(x|z_1,\ldots,z_N).$$

Using lemma 4.1 we have

$$\Delta^{(nl)}(x_1,\ldots,x_n|a',a|b',a\tau^2)$$
$$=\prod_{\mu=1}^{n}(x_\mu-a\tau)\ \det(A^{(n-1l-1)}_\lambda(x_\mu|a'|b')-a\tau^3A^{(n-1l-1)}_{\lambda-1}(x_\mu|a'|b'))_{1\leqslant\lambda,\mu\leqslant n}$$

where we employ the abbreviations $a'=(a_1,\ldots,a_{n-1})$ and $b'=(b_1,\ldots,b_{l-1})$. We multiply the λth row by $a\tau^3$ and add it to the $(\lambda+1)$th row successively for $\lambda=1,\ldots,n-1$. Then we obtain (4.4). □

In terms of $\Delta^{(nl)}$ we have

$$\text{RHS of (2.11)} = \prod_{j=2}^{n+1}(z_j - z_1\tau^2)^{-1}\prod_{j=2}^{n+1}\prod_{i=n+2}^{N}(z_i - z_j\tau^2)^{-1}\prod_{\mu=1}^{n}\oint_C \mathrm{d}x_\mu$$
$$\times\,\hat{\Delta}^{(nl)}(x_1,\ldots,x_n|z_1|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N)\Psi(x_1,\ldots,x_n|z_1,\ldots,z_N) \tag{4.5}$$

where

$$\hat{\Delta}^{(nl)}(x_1,\ldots,x_n|z_1|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N)$$
$$= \prod_{j=2}^{n+1}\frac{z_j - z_1\tau^2}{(z_1 - z_j)\tau}\Delta^{(nl)}(x_1,\ldots,x_n|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1)$$
$$-\sum_{j=2}^{n+1}\frac{(1-\tau^2)z_1}{(z_1 - z_j)\tau}\prod_{\substack{k=2\\k\neq j}}^{n+1}\frac{z_k - z_1\tau^2}{(z_j - z_k)\tau}\prod_{i=n+2}^{N}\frac{z_i - z_j\tau^2}{z_i - z_1\tau^2}$$
$$\times\,\Delta^{(nl)}(x_1,\ldots,x_n|z_1,z_2,\overset{j}{\hat{\ldots}},z_{n+1}|z_{n+2},\ldots,z_N,z_j). \tag{4.6}$$

By repeating the argument of lemma 3.2, we find for $H = H^{(nl)}$

$$H^{(nl)}(z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4) = \prod_{j=2}^{n+1}(z_1\tau^4 - z_j\tau^2)^{-1}\prod_{j=2}^{n+1}\prod_{i=n+2}^{N}(z_i - z_j\tau^2)^{-1}$$
$$\times\, I^{(nl)}(z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4) \tag{4.7}$$

where

$$I^{(nl)}(z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4)$$
$$= \prod_{\mu=1}^{n}\oint_C \mathrm{d}x_\mu\ \det(\widetilde{A}_\lambda^{(nl)}(x_\mu|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4))_{1\leqslant\lambda,\mu\leqslant n}$$
$$\times\,\Psi(x_1,\ldots,x_n|z_1,\ldots,z_N) \tag{4.8}$$

and

$$\widetilde{A}_\lambda^{(nl)}(x|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4) = \frac{x - z_1\tau}{x - z_1\tau^3}A_\lambda^{(nl)}(x|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4)$$
$$+\frac{z_1(-\tau)^{4-N}}{x}\left\{\frac{x - z_1\tau}{x - z_1\tau^3}\prod_{j=2}^{N}\frac{x - z_j\tau}{z_j - z_1\tau^2} - \tau^{2(N-2)}\prod_{j=2}^{N}\frac{x - z_j\tau^{-1}}{z_j - z_1\tau^2}\right\}$$
$$\times\,A_\lambda^{(nl)}(z_1\tau^3|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4). \tag{4.9}$$

Let us prove (2.11) and (2.12) for $H^{(nl)}$. The first equation (2.11) is satisfied if the following proposition holds.

Proposition 4.3. Let $\widetilde{A}_\lambda^{(nl)}$ and $\hat{\Delta}^{(nl)}$ be defined by (4.9) and (4.6), respectively. Then they satisfy the following equation:

$$\det(\widetilde{A}_\lambda^{(nl)}(x_\mu|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4))_{1\leqslant\lambda,\mu\leqslant n}$$
$$= (-\tau)^n\hat{\Delta}^{(nl)}(x_1,\ldots,x_n|z_1|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N). \tag{4.10}$$

Proof. In this proof, we use the abbreviations $x = (x_1,\ldots,x_n)$, $\overset{\nu}{\hat{x}} = (x_1,\overset{\nu}{\hat{\ldots}},x_n)$, $z' = (z_2,\ldots,z_{n+1})$, $\overset{j}{\hat{z}'} = (z_2,\overset{j}{\hat{\ldots}},z_{n+1})$, $z'' = (z_{n+2},\ldots,z_N)$ and $\overset{i}{\hat{z}''} = (z_{n+2},\overset{i}{\hat{\ldots}},z_N)$. First of all, observe

$$\widetilde{A}_\lambda^{(nl)}(x|z'|z'',z_1\tau^4) = \frac{x-z_1\tau}{x-z_1\tau^3}A_\lambda^{(nl)}(x|z'|z'',z_1\tau^4) + g(x|z_1|z'|z'')f_\lambda^{(nl)}(z_1\tau^3|z'|z'',z_1\tau^4)$$

where

$$g(x|z_1|z'|z'')$$
$$:= \frac{z_1(-\tau)^{4-l}}{x}\left\{\frac{x-z_1\tau}{x-z_1\tau^3}\prod_{j=2}^{N}(x-z_j\tau)-\tau^{2(N-2)}\prod_{j=2}^{N}(x-z_j\tau^{-1})\right\}\Big/\prod_{i=n+2}^{N}(z_i-z_1\tau^2).$$

Thanks to the n-fold linearity of the determinant we have

$$\det(\widetilde{A}_\lambda^{(nl)}(x_\mu|z_2,\ldots,z_{n+1}|z_{n+2},\ldots,z_N,z_1\tau^4))_{1\leqslant\lambda,\mu\leqslant n}$$
$$= \prod_{\mu=1}^{n}\frac{x_\mu-z_1\tau}{x_\mu-z_1\tau^3}\det(A_\lambda^{(nl)}(x_\mu|z'|z'',z_1\tau^4))_{1\leqslant\lambda,\mu\leqslant n}$$
$$+\prod_{\mu=1}^{n}(x_\mu-z_1\tau)\sum_{\nu=1}^{n}\prod_{\substack{\mu=1\\ \mu\neq\nu}}^{n}\frac{g(x_\nu|z_1|z'|z'')}{x_\nu-z_1\tau^3}\det(B_{\lambda\mu}^{(nl)})_{1\leqslant\lambda,\mu\leqslant n} \tag{4.11}$$

where

$$B_{\lambda\mu}^{(nl)} = \begin{cases} A_\lambda^{(nl)}(x_\mu|z'|z'',z_1\tau^4)/(x_\mu-z_1\tau^3) & \text{if } \lambda\neq\nu\\ f_\lambda^{(nl)}(z_1\tau^3|z'|z'',z_1\tau^4) & \text{if } \lambda=\nu.\end{cases} \tag{4.12}$$

Both sides of (4.10) are anti-symmetric polynomials in the variables $(x_1,\ldots,x_n)$ and symmetric rational functions in the variables $(z_2,\ldots,z_{n+1})$ and $(z_{n+2},\ldots,z_N)$, respectively. Their homogeneous degree is $n(l-1)+n(n-1)/2$. From (4.6) and (4.11), (4.12) all the singularities come from simple poles located at $z_i = z_1\tau^2$ $(n+1\leqslant i\leqslant N)$.

Let $\mathrm{Cyc}(n,l)$ denote the statement that equation (4.10) holds for (n,l) satisfying $n<l$. Then, in order to verify $\mathrm{Cyc}(n,l)$ it is enough to show the following three claims.

Claim 1: When $z_j = z_1\tau^2$ $(2\leqslant j\leqslant n+1)$, (4.10) holds.

Claim 2: When $z_i = z_j\tau^2$ $(2\leqslant j\leqslant n+1<i\leqslant N)$, (4.10) holds.

Claim 3: The residues at $z_i = z_1\tau^2$ $(n+2\leqslant i\leqslant N)$ in both sides of (4.10) are equal. This sufficiency is based on elementary algebra. If these three are valid, the difference between the two sides is a polynomial of degree $n(l-1)+n(n-1)/2$ because of claim 3; it has $nl+n(n-1)/2$ zeros because of the first two claims and the anti-symmetry with respect to the x_μs and consequently it should vanish. Let us show Cyc(n, l) by induction. We have proved Cyc$(1, l)$ in section 3.

For $n < l$ consider (4.10) under the restriction of $z_j = z_1\tau^2$ $(2 \leqslant j \leqslant n+1)$. Then we get

$$\mathrm{RHS}|_{z_j=z_1\tau^2} = \prod_{i=n+2}^{N} \frac{z_i - z_1\tau^4}{z_i - z_1\tau^2} \Delta^{(nl)}(x|z_1, \overset{j}{\hat{z}'}|z'', z_j)|_{z_j=z_1\tau^2}.$$

In the LHS, we use the recursion relations of $A_\lambda^{(nl)}$ and $f_\lambda^{(nl)}$ and subtract the terms $A_{\lambda-1}^{(n-1l-1)}$ and $f_{\lambda-1}^{(n-1l-1)}$ appearing in the λth row, to find

$$\mathrm{LHS}|_{z_j=z_1\tau^2} = \prod_{\mu=1}^{n}(x_\mu - z_1\tau)\det(A_\lambda^{(n-1l-1)}(x_\mu|\overset{j}{\hat{z}'}|z'') + \tilde{g}(x_\mu) f_\lambda^{(n-1l-1)}(z_1\tau^3|\overset{j}{\hat{z}'}|z''))_{1\leqslant\lambda,\mu\leqslant n}$$

where

$$\tilde{g}(x) = z_1(-\tau)^{4-l} h^{(N-2)}(x|\overset{j}{\hat{z}'}, z'') \Big/ \prod_{i=n+2}^{N}(z_i - z_1\tau^2).$$

Since the nth row is equal to $h^{(N-2)}(x_\mu|\overset{j}{\hat{z}'}, z'')\prod_{i=n+2}^{N}(z_i - z_1\tau^4)/(z_i - z_1\tau^2)$, the second term $\tilde{g}(x_\mu) f_\lambda^{(nl)}(z_1\tau^3|\overset{j}{\hat{z}'}|z'')$ in the λth row can be removed for $\lambda \neq n$. Hence, we obtain

$$\mathrm{LHS}|_{z_j=z_1\tau^2} = \prod_{\mu=1}^{n}(x_\mu - z_1\tau)\prod_{i=n+2}^{N}\frac{z_i - z_1\tau^4}{z_i - z_1\tau^2}\sum_{\nu=1}^{n}(-1)^{n+\nu} h^{(N-2)}(x_\nu|\overset{j}{\hat{z}'}|z'')\Delta^{(n-1l-1)}(\overset{\nu}{\hat{x}}|\overset{j}{z'}|z'').$$

Thus claim 1 is proved.

Next let us show claim 2. Suppose Cyc$(n-1, l-1)$. The value of the RHS at $z_i = z_j\tau^2 (2 \leqslant j \leqslant n+1 < i \leqslant N)$ is given recursively by

$$\mathrm{RHS}|_{z_i=z_j\tau^2} = (-\tau)^{n-1}\frac{z_j - z_1\tau^2}{z_j - z_1}\prod_{\mu=1}^{n}(x_\mu - z_j\tau)\sum_{\nu=1}^{n}(-1)^{n+\nu}$$
$$\times h^{(N-2)}(x_\nu|z_1, \overset{j}{\hat{z}'}, \overset{i}{\hat{z}''})\tilde{\Delta}^{(n-1l-1)}(\overset{\nu}{\hat{x}}, |z_1|\overset{j}{\hat{z}'}|\overset{i}{\hat{z}''}).$$

Repeating a similar calculation to that given before, we have

$$\mathrm{LHS}|_{z_i=z_j\tau^2} = \frac{z_j - z_1\tau^2}{z_j - z_1}\prod_{\mu=1}^{n}(x_\mu - z_j\tau)\sum_{\nu=1}^{n}(-1)^{n+\nu}h^{(N-2)}(x_\nu|z_1, \overset{j}{\hat{z}'}, \overset{i}{\hat{z}''})$$
$$\times \det(\widetilde{A}_\lambda^{(n-1l-1)}(x_\mu|\overset{j}{\hat{z}'}|\overset{i}{\hat{z}''}, z_1\tau^4))_{\substack{1\leqslant\lambda\leqslant n-1 \\ 1\leqslant\mu(\neq\nu)\leqslant n}}.$$

Thus claim 2 follows from the assumption of induction.

Let us turn to claim 3. It follows from (4.6) that the residue of the RHS at $z_i = z_1\tau^2$ is given by

$$(-1)^{n+1} z_1\tau^2(1-\tau^2)\prod_{\mu=1}^{n}(x_\mu - z_1\tau)\sum_{j=2}^{n+1}\prod_{\substack{k=2\\k\neq j}}^{n+1}\frac{z_k - z_1\tau^2}{(z_j - z_k)\tau}\prod_{\substack{m=n+2\\m\neq i}}^{N}\frac{z_m - z_j\tau^2}{z_m - z_1\tau^2}$$

$$\times\sum_{\nu=1}^{n}(-1)^{n+\nu}h^{(N-2)}(x_\nu|z',\overset{i}{\hat{z''}})\Delta^{(n-1l-1)}(\overset{\nu}{\hat{x}}\,|\,\overset{j}{\hat{z'}}\,|\,\overset{i}{\hat{z''}},z_j).$$

From (4.11), that of the LHS is equal to

$$z_1(-\tau)^{4-l}\prod_{\mu=1}^{n}(x_\mu - z_1\tau)\sum_{\nu=1}^{n}h^{(N-2)}(x_\nu|z',\overset{i}{\hat{z''}})\det(B'^{(nl)}_{\lambda\mu})_{1\leqslant\lambda,\mu\leqslant n}\Big/\prod_{\substack{m=n+2\\m\neq i}}^{N}(z_m - z_1\tau^2)$$

where $B'^{(nl)}$ is obtained from $B^{(nl)}$ by the elementary transformation

$$B'^{(nl)}_{\lambda\mu} = \begin{cases}\displaystyle\prod_{j=2}^{n+1}(x_\mu - z_j\tau)\frac{f^{(nl)}_\lambda(x_\mu|z'|z'',z_1\tau^4)|_{z_i=z_1\tau^2} - f^{(nl)}_\lambda(z_1\tau^3|z'|z'',z_1\tau^4)|_{z_i=z_1\tau^2}}{x_\mu - z_1\tau^3} & \\ \displaystyle +\tau^{2(l-n+\lambda-1)}(x_\mu - z_1\tau)\prod_{\substack{m=n+2\\m\neq i}}^{N}(x_\mu - z_m\tau^{-1})g^{(n)}_\lambda(x_\mu|z') & \text{if } \mu\neq\nu \\ f^{(nl)}_\lambda(z_1\tau^3|z'|z'',z_1\tau^4)|_{z_i=z_1\tau^2} & \text{if } \mu=\nu.\end{cases} \tag{4.13}$$

Hence claim 3 is equivalent to

$$\sum_{\nu=1}^{n}h^{(N-2)}(x_\nu|z',\overset{i}{\hat{z''}})\det(B'^{(nl)}_{\lambda\mu}) = (-1)^{N-1}\tau^{l-2}(1-\tau^2)\sum_{j=2}^{n+1}\prod_{\substack{k=2\\k\neq j}}^{n+1}\frac{z_k - z_1\tau^2}{(z_j - z_k)\tau}$$

$$\times\prod_{\substack{m=n+2\\m\neq i}}^{N}(z_m - z_j\tau^2)\sum_{\nu=1}^{n}(-1)^{n+\nu}h^{(N-2)}(x_\nu|z',\overset{i}{\hat{z''}})\Delta^{(n-1l-1)}(\overset{\nu}{\hat{x}}\,|\,\overset{j}{\hat{z'}}\,|\,\overset{i}{\hat{z''}},z_j). \tag{4.14}$$

The poles at $x_\mu = z_1\tau^3$ ($1\leqslant\mu\leqslant n$) in the LHS and at $z_j = z_k$ ($2\leqslant j,k\leqslant n+1$) in the RHS are spurious and consequently both sides of (4.14) are polynomials. Note that

$$f^{(nl)}_\lambda(z_1\tau^3|z'|z'',z_1\tau^4)|_{z_i=z_1\tau^2} = (-\tau)^{l-n+\lambda-2}(1-\tau^2)\varphi^{(n\ l-2)}_{\lambda\ l-n+\lambda-2}(z'|\overset{i}{\hat{z''}}) \tag{4.15}$$

is independent of z_1. Because of (4.3), (4.13) and (4.15), the degree with respect to z_1 of the RHS is $n-1$ and that of the LHS is at most $n-1$. Hence, (4.14) is a polynomial equation in z_1 of degree $n-1$. Owing to claim 1, n points $z_1 = z_j\tau^{-2}$ ($2\leqslant j\leqslant n+1$) satisfy (4.14). Thus it holds identically.

Therefore $\mathrm{Cyc}(n,l)$ is verified. □

The second equation (2.12) is satisfied if the following proposition is valid.

Proposition 4.4.

$$\det(\overline{A}_\lambda^{(nl)}(x_\mu|z_1\tau^{-4}, z_2, \dots, z_{n+1}|z_{n+2}, \dots, z_N))_{1\leqslant\lambda,\mu\leqslant n}$$
$$= (-\tau)^{2-l}\overline{\Delta}^{(nl)}(x_1, \dots, x_n|z_2, \dots, z_n|z_{n+1}, \dots, z_N|z_1) \tag{4.16}$$

where

$$\overline{A}_\lambda^{(nl)}(x|z_1\tau^{-4}, z_2, \dots, z_n|z_{n+1}, \dots, z_N) = \frac{x - z_1\tau^{-1}}{x - z_1\tau^{-3}} A_\lambda^{(nl)}(x|z_1\tau^{-4}, z_2, \dots, z_n|z_{n+1}, \dots, z_N)$$
$$+ \frac{z_1(-\tau)^{N-4}}{x}\left\{\frac{x - z_1\tau^{-1}}{x - z_1\tau^{-3}} \prod_{j=2}^{N} \frac{x - z_j\tau^{-1}}{z_j - z_1\tau^{-2}} - \tau^{-2(N-2)} \prod_{j=2}^{N} \frac{x - z_j\tau}{z_j - z_1\tau^{-2}}\right\}$$
$$\times A_\lambda^{(nl)}(z_1\tau^{-3}|z_1\tau^{-4}, z_2, \dots, z_n|z_{n+1}, \dots, z_N)$$

and

$$\overline{\Delta}^{(nl)}(x_1, \dots, x_n|z_2, \dots, z_n|z_{n+1}, \dots, z_N|z_1)$$
$$= \prod_{j=n+1}^{N} \frac{z_j - z_1\tau^{-2}}{(z_1 - z_j)\tau^{-1}} \Delta^{(nl)}(x_1, \dots, x_n|z_1, \dots, z_n|z_{n+1}, \dots, z_N)$$
$$- \sum_{j=n+1}^{N} \frac{(1-\tau^{-2})z_j}{(z_1 - z_j)\tau^{-1}} \prod_{\substack{k=n+1\\k\neq j}}^{N} \frac{z_k - z_1\tau^{-2}}{(z_j - z_k)\tau^{-1}} \prod_{i=2}^{n} \frac{z_i - z_j\tau^{-2}}{z_i - z_1\tau^{-2}}$$
$$\times \Delta^{(nl)}(x_1, \dots, x_n|z_2, \dots, z_n, z_j|z_{n+1}, \overset{j}{\hat{\dots}}, z_N, z_1).$$

Proof. In a similar way as in Proposition 4.3 the following two claims can be shown.

Claim 1′: when $z_j = z_1\tau^{-2}$ $(n+1 \leqslant j \leqslant N)$ both sides of (4.16) coincide.

Claim 2′: when $z_i = z_j\tau^{-2}$ $(2 \leqslant i \leqslant n < j \leqslant N)$ both sides of (4.16) coincide.

From power counting we do not need an analogue of claim 3. In fact, let us factor out the difference product

$$\prod_{\mu<\nu}(x_\mu - x_\nu) \qquad \text{and} \qquad \prod_{j=2}^{n+1}\left((z_j - z_1\tau^{-2}) \prod_{i=n+2}^{N}(z_i - z_j\tau^{-2})\right)$$

from the difference of both sides. Then it is a rational function of homogeneous degree $-n$ with, at most, $n-1$ simple poles located at $z_i = z_1\tau^{-2}$ $(2 \leqslant i \leqslant n)$ and should therefore be zero. □

Now we are in a position to describe the main theorem of the present paper.

Theorem 4.5. For $n < l$, the integral formula given by

$$H(z_1, \dots, z_N) = \prod_{\mu=1}^{n} \oint_C dx_\mu\, F(x_1, \dots, x_n|z_1, \dots, z_N)\Psi(x_1, \dots, x_n|z_1, \dots, z_N)$$

$$F(x_1,\ldots,x_n|z_1,\ldots,z_N)=\frac{\Delta^{(nl)}(x_1,\ldots,x_n|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)}{\prod_{j=1}^n\prod_{i=n+1}^N(z_i-z_j\tau^2)}$$

$$\Delta^{(nl)}(x_1,\ldots,x_n|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)=\det(A_\lambda^{(nl)}(x_\mu|z_1,\ldots,z_n|z_{n+1},\ldots,z_N))_{1\leqslant\lambda,\mu\leqslant n}$$

$$A_\lambda^{(nl)}(x|a_1,\ldots,a_n|b_1,\ldots b_l):=\prod_{j=1}^n(x-a_j\tau)f_\lambda^{(nl)}(x|a_1,\ldots,a_n|b_1,\ldots,b_l)$$
$$+\tau^{2(l-n+\lambda-1)}\prod_{i=1}^l(x-b_i\tau^{-1})g_\lambda^{(n)}(x|a_1,\ldots,a_n)$$

satisfies (2.11) and (2.12) with $(\delta_+,\delta_-)=(\tau^{-n},\tau^{-l})$.

Finally, we give Smirnov's formula for the case $n=l$. The above formula specialized to $n=l$ does not give a solution. We use an $(n-1)$-fold integration and replace $\Delta^{(nn)}$ by

$$\Delta^{(nn)}(x_1,\ldots,x_{n-1}|a_1,\ldots,a_n|b_1,\ldots,b_n)=\det(A_{\lambda+1}^{(nn)}(x_\mu|a_1,\ldots,a_n|b_1,\ldots,b_n))_{1\leqslant\lambda,\mu\leqslant n-1}. \tag{4.17}$$

In this case, $f_\lambda^{(nn)}(x|a_1,\ldots,a_n|b_1,\ldots,b_n)=g_\lambda^{(n)}(x|b_1,\ldots,b_n)$ and $g_1^{(n)}(x|b_1,\ldots,b_n)=0$ and hence

$$A_\lambda^{(nn)}(x|a_1,\ldots,a_n|b_1,\ldots b_n)$$
$$=\prod_{j=1}^n(x-a_j\tau)g_\lambda^{(n)}(x|b_1,\ldots,b_n)+\tau^{2(\lambda-1)}\prod_{i=1}^n(x-b_i\tau^{-1})g_\lambda^{(n)}(x|a_1,\ldots,a_n).$$

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